Kuwait University | Math 101 | Date: July 12, 2003
Dept. of Math. & Comp. Sci. | First Exam | Duration: 75 minutes

Calculators, mobile phones, pagers and all other mobile communication equipment are not allowed

1. Evaluate the following limits, if exist:

   (a) $\lim_{x\to 0} \dfrac{|x|}{\sin 2x}$. (3 pts.)

   (b) $\lim_{x\to \pi} \cos(x - \sin x)$. (3 pts.)

2. Find the vertical and horizontal asymptotes, if any, of

$$f(x) = \frac{x^3 - x}{x^2 - x - 2}.$$ (3 pts.)

3. Let

$$f(x) = \begin{cases} \dfrac{2\sin(x-1)}{x^2-1} & \text{if, } x > 1, \\ A + B\sin\frac{\pi x}{2} & \text{if, } x = 1, \\ 2A + \dfrac{x^3-1}{x-1} & \text{if, } x < 1. \end{cases}$$

   Find the values of the constants A and B, so that f is continuous at $x = 1$. (4 pts.)

4. Show that the graphs of $f(x) = x^5 + 3x^3 + 1$ and $g(x) = 6x^4 - x^3 + 2x - 1$ intersect. (3 pts.)

5. Show that the graph of $f(x) = \dfrac{x^{2/3}}{x-1}$ has a cusp. (3 pts.)

6. Find an equation for the tangent line to the graph of $f(x) = (x+1)^2 \cos x$ at $x = 0$. (3 pts.)

7. Let $y = u^2 - 5u + 1$ and $u = \dfrac{x-1}{x+1}$, find $\dfrac{dy}{dx}$. (3 pts.)

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1. (a) $\lim_{x\to 0^\pm} \frac{|x|}{\sin 2x} = \frac{1}{2}\lim_{2x\to 0^\pm} \frac{\pm 2x}{\sin 2x} = \boxed{\pm\frac{1}{2}} \Rightarrow \lim_{x\to 0} \frac{|x|}{\sin 2x} = DNE.$

   (b) $\lim_{x\to\pi} \cos(x - \sin x) = \cos[\lim_{x\to\pi}(x - \sin x)] = \cos(\pi - 0) = \boxed{-1}$

2. $x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0 \Rightarrow x = 2$ or $x = -1$.

   $\lim_{x\to -1} f(x) = \lim_{x\to -1} \frac{x(x-1)(x+1)}{(x-2)(x+1)} = \boxed{-\frac{2}{3}} \Rightarrow$ No V.A. at $x = -1$.

   $\lim_{x\to 2^\pm} f(x) = \pm\infty \Rightarrow \boxed{x = 2}$ is V.A

   $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x^3(1-\frac{1}{x^2})}{x^2(1-\frac{1}{x}-\frac{2}{x^2})} = \pm\infty \Rightarrow$ No H.A.

3. $f(1) = A + B\sin\frac{\pi}{2} = A + B$,

   $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{2\sin(x-1)}{x^2-1} = \left(\lim_{x\to 1^+} \frac{2}{x+1}\right)\left(\lim_{(x-1)\to 0^+} \frac{\sin(x-1)}{(x-1)}\right) = \boxed{1}$

   $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \left(2A + \frac{x^3-1}{x-1}\right) = \boxed{2A+3}.$

   $\lim_{x\to 1^+} f(x) = f(1) = \lim_{x\to 1^-} f(x) \Rightarrow \boxed{A = -1, B = 2}$

4. Consider $F(x) = f(x) - g(x) = x^5 - 6x^4 + 4x^3 - 2x + 2$. f and g intersect at the zeros of F. Since $F(0) = 2 > 0$ & $F(-1) = -7 < 0$ and F is continuous on $[-1, 0]$, then from the Intermediate Value Theorem there exists $c \in (-1, 0)$ such that $F(c) = 0$. Thus f and g intersect in $(-1, 0)$.

5. $f'(x) = \frac{-x-2}{3x^{\frac{1}{3}}(x-1)^2}$. f is continuous at $x = 0$, $\boxed{\lim_{x\to 0^\pm} f'(x) = \mp\infty} \Rightarrow$ The graph of f has a cusp at $\boxed{x = 0}$.

6. $\boxed{f'(x) = 2(x+1)\cos x - (x+1)^2 \sin x} \Rightarrow$ Slope of tangent line= $\boxed{f'(0) = 2}$ & $\boxed{f(0) = 1} \Rightarrow$ Equation of tangent line at $P(0, 1)$ is: $\boxed{y = 2x + 1}$

7. $\frac{dy}{du} = 2u - 5$ & $\frac{du}{dx} = \frac{2}{(x+1)^2} \Rightarrow$

   $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (2u-5)\frac{2}{(x+1)^2} = \left[2\left(\frac{x-1}{x+1}\right) - 5\right]\frac{2}{(x+1)^2}.$